

Aims: To provide a significantly improved probability distribution for the H -test for periodicity in X-ray and γ -ray arrival times, which is already extensively used by the γ -ray pulsar community. Also, to obtain an analytical probability distribution for stacked test statistics in the case of a search for pulsed emission from an ensemble of pulsars where the significance per pulsar is relatively low, making individual detections insignificant on their own. This information is timely given the recent rapid discovery of new pulsars with the Fermi-LAT γ -ray telescope. **Methods:** Approximately 10^{14} realisations of the H -statistic (H) for random (white) noise is calculated from a random number generator for which the repetition cycle is $\gg 10^{14}$. From these numbers the probability distribution $P(> H)$ is calculated. **Results:** The distribution of H is found to be exponential with parameter $\lambda = 0.4$ so that the cumulative probability distribution $P(> H) = \exp(-\lambda H)$. If we stack independent values for H , the sum of K such values would follow the Erlang-K distribution with parameter λ for which the cumulative probability distribution is also a simple analytical expression. **Conclusion:**

Searches for weak pulsars with unknown pulse profile shapes in the Fermi-LAT, Agile or other X-ray data bases should benefit from the H -test since it is known to be powerful against a broad range of pulse profiles, which introduces only a single statistical trial if only the H -test is used. The new probability distribution presented here favours the detection of weaker pulsars in terms of an improved sensitivity relative to the previously known distribution.

Key words. methods: statistical – pulsars: general –

The H-test probability distribution revisited: Improved sensitivity

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Abstract.

1. Introduction

When searching for a periodic signal in X-ray or γ -ray arrival times dominated by noise, we may either perform a blind search for γ -ray pulsars as demonstrated by the Fermi-LAT Collaboration (Abdo et al. 2009), or, search for such a signal where the frequency parameters have been prescribed by contemporary radio data (Weltevrede 2010). Following the folding of say N arrival times t_1, \dots, t_N modulo the pulsar spin parameters, we arrive at a set of phases θ_i , $i = 1, \dots, N$. However, in the case of blind searches, Atwood et al. 2006 introduced a time differencing technique in which case the number of trial periods is significantly reduced.

de Jager, Swanepoel & Raubenheimer (1989, hereafter DSR) reviewed the general class of *Beran* statistics (Beran 1969), from which the most general test statistics such as Pearson's χ^2 , Rayleigh and Z_m^2 statistics are derived, and from within this class they derived the well known *H-test* for X-ray and γ -ray Astronomy.

The probability distribution of the *H-test* statistic as given by DSR was derived from Monte Carlo simulations employing $\sim 10^8$ simulations. The computational power and random number simulators on typical IBM machines during the 1980's had limited ranges of applicability and the *H-test* suffered accordingly. For values of $H < 23$ we found that the probability distribution was exponential with parameter $\lambda = 0.398$ (or 0.4), whereas a hard tail developed for $H > 23$, which resulted in a significant compromise in sensitivity.

The old version of the *H-test* probability distribution is already extensively used by e.g. the Fermi-LAT Collaboration for pulsar searches (e.g. Abdo et al. 2009 and Weltevrede et al. 2010), and from this paper it will become clear that the significances assigned to pulsar detections (or non-detections) may

be too conservative, so that some pulsars may be missed, especially when many trial periods are involved, so that large values of the *H*-statistic are required for a significant detection. In this Letter we notify the community that all previous published significances from the *H-test* should be reassessed, based on the new improved distribution presented below. Before we do so, we first briefly review the origin and properties of the *H-test*.

2. The Beran class of test statistics - towards the H-test

Let θ be the pulsar phase measured on the interval $[0, 2\pi)$, so that a full rotation corresponds to 2π . Assuming noise (e.g. from cosmic rays) are also present such that the pulsed fraction is $p \leq 1$. Let $f_s(\theta)$ be the observed line-of-sight pulse profile in the absence of noise. The case $p = 0$ then corresponds to no signal (pure noise), whereas $p = 1$ corresponds to no noise (pure pulsed signal). The observed light curve $f(\theta)$ can then be represented as a mixing of the noise and signal distributions (see Eqn (2) of DSR 1989)

$$f(\theta) = pf_s(\theta) + \frac{(1-p)}{2\pi} \quad (1)$$

The general form of the Beran statistic is given by Beran (1969) in the form (see also DSR 1989)

$$\psi(f) = \int_0^{2\pi} (f(\theta) - \frac{1}{2\pi})^2 d\theta = p^2 \int_0^{2\pi} (f_s(\theta) - \frac{1}{2\pi})^2 d\theta, \quad (2)$$

It is thus clear that the *Beran* statistic measures the integrated squared distance between the pulse profile and uniformity, so that if $p \rightarrow 0$, then $\psi \rightarrow 0$ as well, or, if $f_s = 1/2\pi$ (i.e. a flat uniform distribution), then $\psi = 0$ as well. Thus, we would reject uniformity if ψ exceeds a chosen positive critical value.

It was noted by DSR that by replacing $f(\theta)$ with a density estimator $\hat{f}(\theta)$ based on the observed folded phases θ_i , we retrieve test statistics specific to the kernel of the density estimator. Selecting the Fourier series estimator \hat{f}_m with m harmonics (see Eqn (5) of DSR) as representative of the light curve, results in the well-known Z_m^2 given the proper normalisation (Eqn (7) of DSR)

$$Z_m^2 = 2\pi N \psi(\hat{f}_m). \quad (3)$$

Since $\psi(f)$ scales with p^2 (Eqn 2), it is then clear that the statistic Z_m^2 should scale as $p^2 N$. The quantity $X = p \sqrt{N}$ is then also the approximate Gaussian significance of the point source on the skymap if the background level is well known.

The main problem raised by DSR is that we do not know a-priori the optimal number of harmonics to select. In the case of the popular Z_m^2 test, the optimal number of harmonics would depend on X and the pulse profile shape.

The rationale behind the H -test was to find a consistent estimator for $f(\theta)$ where the number of harmonics m is not chosen subjectively by the observer, since selecting a number of m values for Z_m^2 , until a pleasing result is obtained, may lead to a false detection, given the difficulty to keep track of the number of trials involved.

Hart (1985) derived a technique to obtain the optimal number of harmonics M such that the mean integrated squared error ($MISE$) between the Fourier series estimator $\hat{f}_M(\theta)$ and the true unknown light curve shape is a minimum (see DSR for a review). Thus, $\hat{f}_M(\theta)$ would give the best Fourier series representation of the true pulse profile, given the constraints imposed by the statistics and inherent pulsed fraction. Since the minimum of the $MISE$ involves the finding of a maximum quantity over all harmonics m , DSR redefined this maximal (optimal) quantity in terms of the so-called H -test statistic (named after Hart):

$$H \equiv \max_{1 \leq m \leq 20} (Z_m^2 - 4m + 4) = Z_M^2 - 4M + 4 \geq 0. \quad (4)$$

Uniformity would correspond to low values of M and also low values of H , whereas large values of Z_M^2 (relatively strong pulsed signal) corresponds to large values of H . Subtracting $4M$ and adding 4 (to ensure positive values of H only) effectively limits the variance of H in the absence of a pulsed signal. It is clear that the H test represents a rescaled version of the Z_m^2 test: if $M = 1$, then we retrieve the well-known *Rayleigh test*. DSR have shown that this test is powerful against a wide range of alternatives, which makes this test attractive if the pulse profile shape is not known *a priori*.

To give an example of how H and M depend on the above-mentioned significance $X = p \sqrt{N}$ given real data, we selected archival photon arrival times from the public Fermi-LAT data base from the Vela pulsar direction, with events selected according to the Fermi-LAT point spread function, but with energies > 500 MeV. Folding these phases with the given contemporary spin parameters of Vela reveals a pulsed fraction of $p \sim 0.96$, i.e. nearly 100% and from $N = 600$ events (4 days integration) we already obtain $M = 20$. By adding randomly generated events (i.e. uniformly distributed pulse phases) to the signal events we effectively reduce p and hence X , so that H

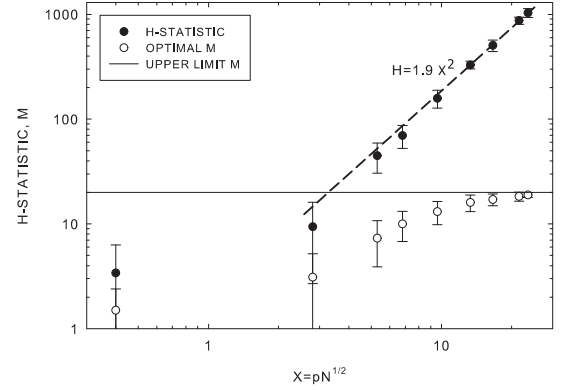


Fig. 1. The dependence of H and M on the approximate skymap significance X for γ -ray pulse phases above 500 MeV from the Vela pulsar as described in the text. The errors represent standard deviations based on 20 independent Fermi-LAT Vela pulsar data sets, each of length up to ~ 4 days. The horizontal line represent the upper limit of $M = 20$, whereas the dashed line represent a fit of the form $H = 1.9X^2$ for $X > 3$.

and M should also decrease accordingly. Figure 1 shows how H and M relates to X : For $X > 3$ (stronger than a $\sim 3\sigma$ signal) we see that H scales with X^2 , with $H = 1.9X^2$, whereas the optimal M is already > 10 for $X > 7$.

This figure is also quite useful to see what typical H and M values we may expect for a typical Vela-like pulsar if we know the strength of the signal as derived from a point source on the skymap, and assuming the excess is due to such a pulsed signal.

3. The revised probability distribution of H for uniformity

The calculations were performed on the Institutional Cluster of the North-West University, Potchefstroom campus. To parallelize the computations, we used the RngStream package (L'Ecuyer 2002), which guarantees independent, non-overlapping substreams of random numbers. The repetition cycle for this random number generator is 3×10^{57} , which is certainly large enough for our purposes. A total of 4×10^{14} samples were calculated in this way.

In the case of large statistics ($N > 100$) we do not need to simulate individual arrival times directly (see the approximate correction factors in Figures 1 and 2 of DSR in the case of low statistics - $N < 100$), so that we only need to simulate the Z_m^2 statistics directly: Since Z_m^2 is the sum of $m \chi_2^2$ -statistics, we can simulate a χ_2^2 statistic directly from a uniformly distributed random number $r \in [0, 1)$ by taking the transformation $-2 \ln r$ and adding these numbers to give Z_m^2 . This speeds up the process considerably. A total of 4×10^{14} values of H were simulated in this way and the results are shown in Fig. 2. The distribution is everywhere consistent with an exponential distribution with parameter $\lambda = 0.398$ (or 0.4), except for $H > 70$ where a downturn relative to the 0.4 index is possibly seen. DSR arrived at the same precise value of $\lambda = 0.398$ for small values of H (i.e. less than 23) since the random number generator used

by DSR did not yet reach the limit of its random cycle for the number of simulations required to reach $H \sim 23$ (with a relatively small error on the corresponding probability) and should therefore reveal the same result as ours for $H < 23$.

It is thus clear that the probability distribution of the H -statistic can be conservatively described by the simple formula

$$\text{Prob}(> H) = \exp(-0.4H). \quad (5)$$

4. Incoherent Stacking

Suppose we analysed the data from K pulsars, or, K independent observations of the same pulsar, where the effect of e.g. unrecorded glitches excluded the possibility of analysing all the data as one single coherent set of arrival times. In this case we want to see if there is a net signal represented by K values of the H -statistic. Suppose we arrive at a set of K such values of H , given by H_i , $i = 1, \dots, K$.

Since we have shown (to the probability level of $\sim 10^{-14}$) that the H -statistic follows that of an exponential distribution with parameter $\lambda = 0.4$, we can stack these test statistics through the sum

$$H_T = \sum_{i=1}^K H_i, \quad (6)$$

which is known to follow the Erlang- K distribution with parameter λ , so that the significance (or probability for uniformity) of such stacking is given by the simple analytical expression (see e.g., Leemis & McQueston 2008 on univariate distribution relationships, which includes the Erlang distribution as the sum of independent exponential variables with parameter λ .)

$$P(> H_T | K, \lambda) = \sum_{n=0}^{K-1} \frac{\exp(-\lambda H_T) (\lambda H_T)^n}{n!}. \quad (7)$$

5. Conclusions

For $M = 1$ we retrieve the well-known *Rayleigh test*, with the exception that the parameter for the exponential distribution has been reduced from $\lambda = 0.5$ (for the *Rayleigh test*) to $\lambda = 0.4$ for the H -test. This slight loss of sensitivity is the effect of the number of trials taken implicitly into account as a result of the search through m within the H -test.

Finally, it is clear that the corrected distribution of the H -statistic follows a simple exponential with parameter $\lambda = 0.4$ and evaluation of results for $H > 23$ (i.e. the 10^{-4} significance level) would yield more significant results compared to the old distribution presented by DSR. For example, for $H = 50$ a probability of 4×10^{-8} is typically quoted in the literature, whereas the true probability for uniformity is actually 2×10^{-9} - already a factor 20 smaller.

A Fermi-LAT example of the Vela pulsar (> 500 MeV) shows clearly values for M up to 20 for “skymap” significances $X = p\sqrt{N} > 20$, whereas H scales with $H \propto X^2$ as expected for Beran-type tests. The scaling $H = 1.9X^2$ can be used to predict H -test statistics for Vela-like pulsars above 500 MeV if we assume the excess on the skymap is all pulsed.

The hard tail of the distribution beyond $H > 23$ presented by DSR probably arose from the repetition cycle of the random

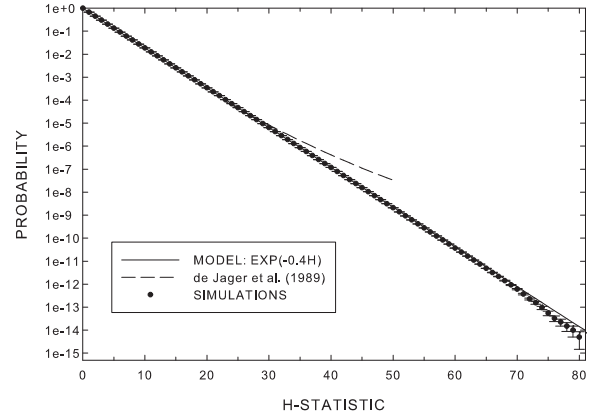


Fig. 2. The distribution of the H -statistic derived from 4×10^{14} Monte Carlo simulations with the best fit model for the cumulative probability $P(> H) = \exp(-0.4H)$ (solid line) and the old version of the probability distribution given by DSR shown as a dashed line.

number generator used in those days, so that the same fluctuations at large H values were repeated given the finite cycle length of the generator used. In this case we however used a generator with a cycle time much longer than 10^{14} , in which case we did not see the repetition of outliers as a result of a finite cycle length. Confirmation of the possible break (i.e. downturn) in the probability distribution at $H > 70$ requires extensive simulations beyond 4×10^{14} and is beyond the scope of this paper.

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References

- Abdo, A.A. et al. 2009, *Science*, 325, (5942), 840
- Atwood, W.B., Ziegler, M., Johnson, R.P., Baughman, B.M. 2006, *ApJ*, 652, L49.
- Weltevrede, P., Abdo, A.A., et al. 2010, *ApJ*, 708, 1426
- Beran, R.J. 1969, *Ann. Math. Stat.*, 40, 1196
- Buccheri, R., & de Jager, O.C. 1989, *NATO A.S.I. Workshop on Timing Neutron Stars*, eds. H. Ögelman & E.P.J. van den Heuvel, Kluwer, Dordrecht, p. 95
- de Jager, O.C., Swanepoel, J.W.H., & Raubenheimer, B.C. 1989, *A&A*, 221, 180 (DSR)
- Hart, J.D. 1985, *J. Statist. Comput. Simul.*, 21, 95
- Leemis, L.M. & McQueston, J.T. 2008, *American Statistician*, 62, 45
- L'Ecuyer, P., Simard, R., Chen, E. J., Kelton, W. D. 2002, *An Object-Oriented Random-Number Package With Many Long Streams and Substreams*, *Operations Research* 50 (6)